# Magnetohydrodynamic flow in a rectangular tube at high Hartmann number

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Asymptotic forms, valid for high Hartmann number $M$, are obtained for the mean velocity for laminar magnetohydrodynamic flow in a rectangular tube. For a tube with non-conducting walls it is found that, neglecting exponentially damped terms, the mean velocity can be expressed in closed form in terms of tabulated functions. The first three terms of the expansion of the mean velocity in inverse powers of $M$ are in extremely close agreement with a corresponding expansion obtained by Shercliff (1953) using a boundary-layer method.

For perfectly conducting walls the first five terms of an expansion in inverse powers of $M$ are obtained.

## 1. Introduction

The mathematical problems encountered in the theory of steady magnetohydrodynamic flow through tubes are such that, under certain conditions, exact solutions can be obtained in series form by the method of separation of variables. Such solutions have been obtained for a rectangular tube by Shercliff (1953) for the case when the tube walls are non-conducting and by Chang & Lundgren (1961) for a tube with perfectly conducting walls. A similar type of exact solution for a circular tube with non-conducting walls has been obtained by Gold (1962) and others (e.g. Uflyand 1961).

The above solutions are not particularly suitable for obtaining information about the solutions for large values of the Hartmann number $M$ and in order to obtain information of this type it seems simpler to use approximate methods rather than attempt to obtain the asymptotic expansion of an exact solution. For a non-conducting rectangular duct the first three terms in an asymptotic expansion of the mean velocity for large $M$ was obtained by Shercliff (1953) by means of a boundary-layer method. He also obtained the first term of such an expansion for a tube of arbitrary cross-section. Chang & Lundgren (1961) generalized Shercliff's work to obtain the corresponding first term for a thin-walled tube of arbitrary conductivity and cross-section. Shercliff (1962) has also generalized his earlier work to give the first two terms in the expansion of the mean velocity for tubes of arbitrary cross-section (but excluding rectangular ones) with non-conducting walls. An alternative approximate method for tubes of rectangular cross-section has been suggested by Grinberg (1961). It is somewhat involved and he does not, in fact, obtain a complete explicit solution.

For tubes of arbitrary cross-section it is clear that it will be necessary to employ approximate methods of solution, but it seems very difficult to obtain more than two or three terms of an asymptotic expansion in this manner. It therefore seems to be both of practical use and theoretical interest to examine whether it is possible to obtain asymptotic expansions for large $M$ of some of the known exact solutions. The only case where this has been considered seems to be in the work of Gold (1962) who obtains the first two terms in an asymptotic expansion of the velocity in a circular tube for high $M$. The resulting expression is singular at two points on the tube surface but integration to determine the mean velocity produces a finite result.

The present note considers the form for high $M$ of the expressions for the mean velocity derived by Shercliff (1953) and Chang & Lundgren (1961) for a rectangular tube with non-conducting and conducting walls, respectively. The general method of approach is to use various integral representations to reduce the expression for the mean velocity to an integral which can be evaluated asymptotically by standard methods. For the problem considered by Shercliff it is found that, neglecting terms which are exponentially damped for high $M$, the mean velocity can be written in a closed form involving tabulated functions. It has not been possible to obtain a similar simple result for the case of conducting walls, but it is possible to obtain as many terms of an asymptotic expansion as one wishes. In the present case five terms are obtained. For both cases the resulting expansions are accurate to within about 5 % for $M = 10$.

## 2. Non-conducting walls

The exact solution for steady magnetohydrodynamic flow in a rectangular tube with non-conducting walls was first obtained by Shercliff (1953). Shercliff found that the mean velocity, $v_0$, was given by

$$v_0 = \frac{32kb^2}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \left\{1 - \frac{2Nb^2(\cosh N - \cosh M)}{(2n+1)^2 \pi^2 a^2 \sinh N}\right\}, \tag{1}$$

where $N^2 = M^2 + (2n+1)\pi^2 a^2/b^2$, $M$ is the Hartmann number, $k$ is a constant determined by the pressure gradient and $2a$, $2b$, represent the lengths of the sides of the tube. The applied magnetic field is in the direction of the sides of length $2a$. In the following analysis the dimensionless parameter $\pi a/b$ will be denoted by $\lambda$.

We have, on neglecting terms which are exponentially damped for large $M$, that

$$v_0 \sim \frac{32kb^2}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} + \frac{64kb^2}{\pi^4 \lambda^2} I, \tag{2}$$

where

$$I = \sum_{n=0}^{\infty} \frac{(e^{M-N} - 1)\,N}{(2n+1)^6}. \tag{3}$$

It follows from the identity (Watson 1944, p. 146)

$$\int_0^{\infty} \frac{\mathrm{J}_0(\beta t) \exp\left[-\alpha(t^2+y^2)^{\frac{1}{2}}\right]}{(t^2+y^2)^{\frac{1}{2}}}\, t\,dt = \exp\frac{\left[-y(\alpha^2+\beta^2)^{\frac{1}{2}}\right]}{(\alpha^2+\beta^2)^{\frac{1}{2}}}, \tag{4}$$

that

$$I = M^2(L_3 - N_3) + \lambda^2(L_2 - N_2), \tag{5}$$

where $$L_r = e^M \int_0^\infty \frac{J_0(Mt)\, t g_r[(1+t^2)^{\frac{1}{2}}]\, dt}{(1+t^2)^{\frac{1}{2}}} \quad (r = 2, 3), \tag{6}$$

$$N_r = \int_0^\infty J_0(Mt)\, g_r(t)\, dt \quad (r = 2, 3) \tag{7}$$

and $$g_r(z) = \sum_{n=0}^{\infty} \frac{e^{-\lambda(2n+1)z}}{(2n+1)^{2r}}. \tag{8}$$

The relationship $$J_0(z) = \tfrac{1}{2}[H_0^{(1)}(z) - H_0^{(1)}(z e^{i\pi})] \tag{9}$$
enables equation (6) to be re-written as

$$L_r = \tfrac{1}{2} e^M \int_{-\infty}^{\infty} \frac{H_0^{(1)}(Mt)\, t g_r[(1+t^2)^{\frac{1}{2}}]\, dt}{(1+t^2)^{\frac{1}{2}}}. \tag{10}$$

The path of integration in equation (10) may be deformed into the upper half of the complex $t$ plane and, in view of the branch point of the integrand at $t = i$, may be replaced by the loop integral around the imaginary axis cut from $i$ to $i\infty$. Hence
$$L_r = \frac{2}{\pi} e^M \int_0^\infty K_0[M(v^2+1)^{\frac{1}{2}}]\, f_r(v)\, dv, \tag{11}$$

where $$f_r(v) = \sum_{n=0}^{\infty} \frac{\cos(2n+1)\lambda v}{(2n+1)^{2r}}$$
$$= Gl_{2r}(\lambda v) - 2^{-2r} Gl_{2r}(2\lambda v). \tag{12}$$

The notation in equation (12) is that suggested by Lewin (1958). In the interval $0 \leqslant \theta \leqslant 2\pi$ the function $Gl_{2r}(\theta)$ can be expressed in terms of the Bernoulli Polynomials $B_{2r}(\theta/2\pi)$ (Lewin 1958) and hence

$$L_r = \frac{2^{2r}\pi^{2r-1}}{(2r)!} (-)^{1+r} e^M \sum_{n=0}^{\infty} \left\{ \int_{2n\pi/\lambda}^{2(n+1)\pi/\lambda} K_0[M(v^2+1)^{\frac{1}{2}}] \times B_{2r}\left(\frac{\lambda v}{2\pi}\right) dv - \frac{1}{2^{2r}} \int_{n\pi/\lambda}^{(n+1)\pi/\lambda} K_0[M(v^2+1)^{\frac{1}{2}}]\, B_{2r}\left(\frac{\lambda v}{\pi}\right) dv \right\}. \tag{13}$$

All the terms in equation (13) with $n > 0$ will be exponentially damped as $M \to \infty$ and hence

$$L_r \sim \frac{2^{2r}\pi^{2r-1}}{(2r)!} (-)^{1+r} e^M \int_0^{2\pi/\lambda} K_0[M(v^2+1)^{\frac{1}{2}}]\, B_{2r}\left(\frac{\lambda v}{2\pi}\right) dv - 2^{-2r} \times \int_0^{\pi/\lambda} K_0[M(v^2+1)^{\frac{1}{2}}]\, B_{2r}\left(\frac{\lambda v}{\pi}\right) dv. \tag{14}$$

The asymptotic forms of the integrals in equation (14) could now be obtained by replacing $K_0$ by its asymptotic expansion, the resulting integrals would then be of the type considered by Erdelyi (1956) and the complete expansion could be obtained by integration by parts. It is simpler, however, to replace the upper limits in the above integrals by infinity, the error involved in this replacement being exponentially small for large $M$. We then have that

$$L_r \sim \frac{2^{2r}\pi^{2r-1}}{(2r)!} (-)^{1+r} e^M \int_0^\infty K_0[M(v^2+1)^{\frac{1}{2}}] \left\{ B_{2r}\left(\frac{\lambda v}{2\pi}\right) - 2^{-2r} B_{2r}\left(\frac{\lambda v}{\pi}\right) \right\} dv. \tag{15}$$

The integrals in equation (15) may be evaluated in closed form (Watson 1944, p. 417), giving

$$L_3 \sim \frac{1}{6!}\left\{\frac{3\pi^6}{4M} - \frac{15\lambda^2\pi^4}{4M^2} + \frac{45\pi^2\lambda^4}{4M^3}\left[1 + \frac{3}{M} + \frac{3}{M^2}\right] - \frac{24\lambda^5}{M^3} e^M \mathrm{K}_3(M)\right\}, \tag{16}$$

$$L_2 \sim \frac{\pi^4}{3\cdot 2^5 M} - \frac{\lambda^2\pi^2}{16M^2}\left(1 + \frac{1}{M}\right) + \frac{\lambda^3}{6M^2} e^M \mathrm{K}_2(M). \tag{17}$$

From equations (7) and (9) it follows that

$$N_r = \frac{1}{2}\int_0^\infty \mathrm{H}_0^{(1)}(Mt)\, g_r(t)\, dt - \frac{1}{2}\int_0^\infty \mathrm{H}_0^{(1)}(Mt\, e^{i\pi})\, g_r(t)\, dt. \tag{18}$$

The behaviour of the Hankel function at infinity shows that the path of integration in the first integral in equation (18) may be replaced by a path along the positive imaginary axis. Similarly the path in the second integral may be replaced by one along the negative imaginary axis. Hence

$$\begin{aligned} N_r &= \frac{1}{\pi}\int_0^\infty \mathrm{K}_0(Mt)\,[g_r(it) + g_r(-it)]\, dt \\ &= \frac{2}{\pi}\int_0^\infty \mathrm{K}_0(Mt)\,[Gl_{2r}(\lambda t) - 2^{-2r} Gl_{2r}(2\lambda t)]\, dt. \end{aligned} \tag{19}$$

The asymptotic evaluation of $N_r$ may now be completed by an analysis similar to that leading to equations (16) and (17), the final result being

$$N_3 \sim \frac{1}{6!}\left\{\frac{3\pi^6}{4M} - \frac{15\pi^4\lambda^2}{4M^3} + \frac{135\lambda^4\pi^2}{4M^5} - \frac{3\cdot 2^6}{M^6}\lambda^5\right\}, \tag{20}$$

$$N_2 \sim \frac{\pi^4}{3\cdot 2^5 M} - \frac{\lambda^2\pi^2}{16M^3} + \frac{\lambda^3}{3M^4}. \tag{21}$$

Equations (2), (5), (16), (17), (20) and (21) now give

$$v_0 \sim \frac{ka^2}{M}\left\{1 - \frac{1}{M} - \frac{32a}{15\pi b M^3} - \frac{32a}{15\pi b} e^M \mathrm{K}_3(M) + \frac{32a}{3\pi b}\frac{e^M \mathrm{K}_2(M)}{M}\right\}. \tag{22}$$

The first terms of the asymptotic expansions of the K functions show that

$$v_0 \sim \frac{ka^2}{M}\left\{1 - \frac{32}{15(2\pi)^{\frac{1}{2}}}\frac{a}{bM^{\frac{1}{2}}} - \frac{1}{M} + \frac{a}{b}\frac{4}{3(2\pi)^{\frac{1}{2}}} M^{-\frac{3}{2}} + O\left(\frac{1}{M^2}\right)\right\}. \tag{23}$$

The value (to four significant digits) of the coefficient of $[a/bM^{\frac{1}{2}}]$ in equation (23) is $-0{\cdot}8511$; the corresponding coefficient obtained by Shercliff by a boundary-layer method is $-0{\cdot}852$.

As the derivation of equation (22) is moderately elaborate it is of interest to note that the approximate equation (23) can be derived in a more elementary, but less rigorous, fashion. This alternative method will now be considered.

Direct expansion of equation (3) shows that

$$I = -M\sum_{n=0}^{\infty}\frac{\{1-\exp[-t(2n+1)^2]\}}{(2n+1)^6} + \tfrac{1}{2}t^2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - t\sum_{n=0}^{\infty}\frac{\{1-\exp[-t(2n+1)^2]\}}{(2n+1)^4} + O\left(\frac{1}{M^3}\right), \tag{24}$$

where $t = \lambda^2/2M$. We also have that

$$\sum_{n=0}^{\infty}\frac{\{1-\exp[-t(2n+1)^2]\}}{(2n+1)^6} = t\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} - \frac{t^2}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \frac{1}{2}\int_0^t (t-s)^2\sum_{n=0}^{\infty}\exp[-s(2n+1)^2]\,ds, \tag{25}$$

$$\sum_{n=0}^{\infty}\frac{\{1-\exp[-t(2n+1)^2]\}}{(2n+1)^4} = t\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \int_0^t (t-s)\sum_{n=0}^{\infty}\exp[-s(2n+1)^2]\,ds. \tag{26}$$

It follows immediately from the Poisson summation formula that

$$\sum_{n=0}^{\infty}\exp[-s(2n+1)^2] \sim \frac{1}{4}\left(\frac{\pi}{s}\right)^{\frac{1}{2}} + \text{terms which tend to zero exponentially as } s\to 0. \tag{27}$$

The approximate equation (23) may now be deduced immediately from equations (2), (24) to (27).

## 3. Perfectly conducting walls

The exact solution for steady magnetohydrodynamic flow in a rectangular tube with perfectly conducting walls was obtained by Chang & Lundgren (1961). From their results, and using the notation of the previous section, it follows that, neglecting terms which are exponentially damped for large $M$,

$$v_0 \sim 2ka^2\sum_{n=0}^{\infty}\frac{1}{\beta_n^2(\beta_n^2+M^2)}\left\{1-\frac{a(\beta_n 2)^{\frac{1}{2}}}{b(\beta_n^2+M^2)^{\frac{1}{2}}[\beta_n+(\beta_n^2+M^2)^{\frac{1}{2}}]^{\frac{1}{2}}}\right\}, \tag{28}$$

where $\beta_n = \frac{1}{2}(2n+1)\pi$.

The first term in equation (28) may be written in a closed form as

$$(ka^2/M^2)(1-M^{-1}\tanh M)$$

and is the exact form for the mean velocity between two parallel conducting planes separated by a distance $2a$. The other term in equation (28) may be written as $-64\,2^{\frac{1}{2}}a^3kS/b\pi^5$ where

$$S = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^{\frac{3}{2}}\{(2n+1)^2+c^2\}^{\frac{1}{2}}[2n+1+\{(2n+1)^2+c^2\}^{\frac{1}{2}}]^{\frac{1}{2}}}$$

and $c = 2M/\pi$. It follows from the identities (Watson 1944, p. 386)

$$\frac{1}{(\alpha^2+\beta^2)^{\frac{1}{2}}[\alpha+(\beta^2+\alpha^2)^{\frac{1}{2}}]^{\frac{1}{2}}} = \frac{1}{\beta}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\frac{e^{-\alpha t}\sin\beta t\,dt}{t^{\frac{1}{2}}} \quad\text{and}\quad \frac{1}{\alpha^2+\beta^2} = \frac{1}{\beta}\int_0^{\infty}e^{-\alpha w}\sin\beta w\,dw$$

that
$$S = \frac{1}{c^2(2\pi)^{\frac{1}{2}}} \int_0^\infty \int_0^\infty \frac{H(w+t)}{t^{\frac{1}{2}}} \{\cos c(t-w) - \cos c(t+w)\}\, dt\, dw, \tag{29}$$
where
$$H(z) = \sum_{n=0}^{\infty} \frac{e^{-z(2n+1)}}{(2n+1)^{\frac{3}{2}}}.$$

$H(z)$ is directly related to the $\Phi$ function considered in Erdelyi (1953) and from equation (8) of this reference it may be shown that
$$H(z) = -(\pi z)^{\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{(-)^n z^n}{n!} \zeta(\tfrac{3}{2}-n)(1-2^{n-\frac{3}{2}}), \tag{30}$$
where $\zeta(s)$ is the Riemann Zeta-function.

$S$ can clearly be written in the form $S_1 - S_2$ where
$$S_1 = \frac{1}{2c^2\sqrt{\pi}} \int_0^\infty H(x) \int_{-x}^{x} \frac{\cos cy\, dy}{(x+y)^{\frac{1}{2}}}\, dx, \tag{31}$$
and
$$S_2 = \frac{1}{2c^2\sqrt{\pi}} \int_0^\infty H(x) \cos cx \int_{-x}^{x} \frac{dy\, dx}{(x+y)^{\frac{1}{2}}}$$
$$= \frac{1}{c^2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty x^{\frac{1}{2}} H(x) \cos cx\, dx.$$

Re-arrangement of equation (31) gives
$$S_1 = \frac{1}{2c^2\sqrt{\pi}} \int_0^\infty H(x) \left\{\left(\frac{\pi}{c}\right)^{\frac{1}{2}} \cos\left(cx - \frac{\pi}{4}\right) - \int_x^\infty \frac{\cos cy\, dy}{(x+y)^{\frac{1}{2}}}\right\} dx, \tag{32}$$
the order of integration in the double integral in equation (32) may now be interchanged and we finally obtain
$$S = \frac{1}{2c^{\frac{5}{2}}} \int_0^\infty H(x) \cos\left(cx - \frac{\pi}{4}\right) dx - \frac{1}{2c^2\sqrt{\pi}} \int_0^\infty \cos cx \left\{2\sqrt{2}x^{\frac{1}{2}} H(x) + \int_0^x \frac{H(y)\, dy}{(x+y)^{\frac{1}{2}}}\right\} dx. \tag{33}$$

The integrals in equation (33) are standard Fourier integrals whose asymptotic expansions may be obtained from the general formulae given by Lighthill (1958) provided that expansions valid near $x = 0$ can be obtained for the functions multiplying the trigonometric ones. The expansions for $H(x)$, $x^{\frac{1}{2}}H(x)$ follow immediately from equation (30) and also from this equation it follows that
$$\int_0^x \frac{H(y)\, dy}{(x+y)^{\frac{1}{2}}} = -\sqrt{\pi}[\sqrt{2} - \log(1+\sqrt{2})]x + \sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2}}(-)^n}{n!} \zeta(\tfrac{3}{2}-n)(1-2^{n-\frac{3}{2}}) \int_0^1 \frac{v^n\, dv}{(1+v)^{\frac{1}{2}}}.$$

Lighthill's general formulae now give
$$S = \frac{1}{c^{\frac{7}{2}}} \zeta\left(\frac{3}{2}\right)\left(1 - \frac{1}{2\sqrt{2}}\right) - \frac{1}{2c^4}[3\sqrt{2} - \log(1+\sqrt{2})] - \frac{1}{2c^{\frac{9}{2}}} \zeta\left(\frac{1}{2}\right)\left(1 - \frac{1}{\sqrt{2}}\right) + O\left(\frac{1}{c^5}\right),$$

the first term of the above expansion could have been deduced immediately from the form of $S$. Finally

$$v_0 = \frac{ka^2}{M^2}\left[1 - \frac{1}{M} - \frac{8a}{b\pi^{\frac{3}{2}} M^{\frac{3}{2}}}\zeta\left(\frac{3}{2}\right)\left(1 - \frac{1}{2\sqrt{2}}\right) + \frac{2\sqrt{2}a}{b\pi M^2}[3\sqrt{2} - \log(1+\sqrt{2})] + \frac{2a}{b\sqrt{\pi}M^{\frac{5}{2}}}\zeta\left(\frac{1}{2}\right)\left(1 - \frac{1}{\sqrt{2}}\right) + O\left(\frac{1}{M^3}\right)\right]$$

$$= \frac{ka^2}{M^2}\left[1 - \frac{1}{M} - \frac{a}{bM^{\frac{3}{2}}}\left\{2{\cdot}43 - \frac{3{\cdot}03}{M^{\frac{1}{2}}} + \frac{0{\cdot}48}{M}\right\} + O\left(\frac{1}{M^3}\right)\right].$$

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